INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 8, 1349–1360 (1988)

EULERIAN–LAGRANGIAN METHODS FOR THE NAVIER–STOKES EQUATIONS AT HIGH REYNOLDS NUMBER

VINCENZO CASULLI

Universita' Degli Studi Di Trento, Dipartimento di Matematica, 38050 Povo (Trento), Italy

SUMMARY

In this paper an explicit Lagrangian approach to advective and diffusive term treatment has been derived to improve the stability and to reduce the artificial diffusion of a finite difference scheme for convection–diffusion equations. This concept is then applied to discretize the convective and viscous terms in the Navier–Stokes equations. The pressure gradient and the velocity divergence are discretized by implicit finite differences in such a way that the resulting velocity field is exactly discrete divergence-free at all times. The stability of the method is shown to become less restrictive as the Reynolds number increases. At large time steps the artificial viscosity also reduces and higher accuracy is obtained. Moreover, the present algorithm is so devised as to take full advantage of vector computations in view of a possible implementation of it on an array computer. The performance of the method is illustrated by the numerical solution obtained for the cavity flow problem at high Reynolds numbers.

KEY WORDS Eulerian–Lagrangian methods Navier–Stokes equations

INTRODUCTION

The unsteady Navier–Stokes equations form a coupled non-linear system of partial differential equations that describes the flow of an incompressible fluid. At high Reynolds number these equations are difficult to solve. Much success has been obtained by efficiently solving the Navier–Stokes equations at low Reynolds numbers, but a great computational effort is still required to solve these equations as the Reynolds number becomes large (see e.g. References 1 and 2).

Fully implicit finite difference methods for solving transient fluid flow problems are often stable for any positive value of the time increment.[3] Implicit methods, however, require the simultaneous solution of a large number of coupled non-linear equations at each time step and consequently they become expensive for three-dimensional flow problems. Alternatively, several semi-implicit finite difference methods have been developed and used (see e.g. Reference 4). These methods are limited by a stability condition on the time step size which easily becomes prohibitively small at high Reynolds number; this is the case if, for example, space-centred finite differences are used to discretize the convective terms. When upwind finite differences are used, the corresponding stability condition is not as restrictive as the one imposed by the use of centred differences, but the artificial viscosity thus introduced by the scheme may be orders of magnitude higher than the physical viscosity.

0271–2091/88/101349–12$06.00

Received 30 September 1987
Revised 5 April 1988

The method which will be introduced in this paper to approximate the solution of Navier–Stokes equations is based on the marker-and-cell method for solving the pressure field[5] and uses a Lagrangian approach on a fixed Eulerian grid system to discretize the convective and viscous terms. Some noteworthy features of this method are as follows:

(a) It utilizes primitive variables, i.e. velocities and pressure rather than streamline and vorticity functions.
(b) The discrete pressures are located at mesh cell centres while the discrete velocities are located on mesh cell sides.
(c) The resulting velocity field is exactly discrete divergence-free at all times.
(d) The stability of the scheme depends only upon the viscosity coefficient: at high Reynolds number a large time step size is allowed.
(e) Artificial viscosity still exists but is brought under control either by reducing the spatial increments or by increasing the time step size.
(f) Most of the required arithmetic operations are independent of each other and highly vectorizable for an efficient implementation on vector computers.

Although this method possesses several important properties, it becomes non-competitive with more classical methods at low Reynolds number, since for such problems better accuracy can be achieved with smaller computational effort.

CENTRED AND UPWIND FINITE DIFFERENCES

Consider the following convection–diffusion equation in two space dimensions:

$$\frac{\partial c}{\partial t}+u\frac{\partial c}{\partial x}+v\frac{\partial c}{\partial y}=\nu\left(\frac{\partial^2 c}{\partial x^2}+\frac{\partial^2 c}{\partial y^2}\right), \tag{1}$$

where, for the time being, the convective coefficients u and v are assumed to be constants and non-negative; the constant ν is a positive diffusion coefficient.

Equation (1) can be solved numerically in a variety of ways. Any explicit finite difference method for equation (1) is of the following form:

$$c_{i,j}^{k+1}=Fc_{i,j}^{k}, \tag{2}$$

where F is a linear difference operator corresponding to the spatial discretization of the convective and diffusive terms. The simplest form for the operator F is obtained when centred finite differences are used to approximate the spatial derivatives in equation (1):

$$\begin{aligned}Fc_{i,j}^{k}=c_{i,j}^{k}&-u\Delta t\frac{c_{i+1,j}^{k}-c_{i-1,j}^{k}}{2\Delta x}-v\Delta t\frac{c_{i,j+1}^{k}-c_{i,j-1}^{k}}{2\Delta y}\\&+\nu\Delta t\left(\frac{c_{i+1,j}^{k}-2c_{i,j}^{k}+c_{i-1,j}^{k}}{\Delta x^2}+\frac{c_{i,j+1}^{k}-2c_{i,j}^{k}+c_{i,j-1}^{k}}{\Delta y^2}\right).\end{aligned} \tag{3}$$

The amplification factor of F is given by

$$f=1-\mathrm{I}[a\sin(\alpha)+b\sin(\beta)]-2\,(d_x+d_y)+2d_x\cos(\alpha)+2d_y\cos(\beta), \tag{4}$$

where $a=u\Delta t/\Delta x$, $b=v\Delta t/\Delta y$ are the Courant numbers, $d_x=\nu\Delta t/\Delta x^2$, $d_y=\nu\Delta t/\Delta y^2$, α and β are the x and y phase angles and $\mathrm{I}=\sqrt{(-1)}$. In order for F to be stable, the time step size must be small

enough so that $|f| \leqslant 1$ uniformly for every α and β. In this case[6] the required necessary and sufficient stability condition is given by

$$\Delta t \leqslant \min\left[\frac{\Delta x^2 \Delta y^2}{2\nu(\Delta x^2+\Delta y^2)}, \frac{2\nu}{u^2+v^2}\right]. \tag{5}$$

Inequality (5) shows that method (2) cannot apply efficiently to convection-dominated problems (i.e. $u^2+v^2 \gg \nu$), since in that case the time step becomes too small for practical calculations.

If first-order upwind finite differences are used for the convective terms and centred differences for the diffusive terms, the operator F takes the form

$$Fc_{i,j}^k = c_{i,j}^k - u\Delta t\frac{c_{i,j}^k - c_{i-1,j}^k}{\Delta x} - v\Delta t\frac{c_{i,j}^k - c_{i,j-1}^k}{\Delta y} + \nu\Delta t\left(\frac{c_{i+1,j}^k - 2c_{i,j}^k + c_{i-1,j}^k}{\Delta x^2} + \frac{c_{i,j+1}^k - 2c_{i,j}^k + c_{i,j-1}^k}{\Delta y^2}\right). \tag{6}$$

In this case the amplification factor of F is given by

$$f = 1 - a[1-\cos(\alpha)+\mathrm{I}\sin(\alpha)] - b[1-\cos(\beta)+\mathrm{I}\sin(\beta)] - 2(d_x+d_y) + 2d_x\cos(\alpha) + 2d_y\cos(\beta). \tag{7}$$

Hence a necessary and sufficient condition to have $|f|<1$ is given by

$$\Delta t \leqslant \min\left[\frac{|u|}{\Delta x}+\frac{|v|}{\Delta y}+2\nu\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)\right]^{-1}. \tag{8}$$

Note that in convection-dominated problems the stability condition (8) is not as restrictive as (5). Now the diffusion coefficient can go to zero and method (2) will remain stable with a non-zero time step size. This method, however, is only first-order accurate in space, and the truncation error has the form of a viscosity term. In fact, when upwind differences are used, a Taylor expansion of each term in (2) yields

$$\frac{\partial c}{\partial t}+u\frac{\partial c}{\partial x}+v\frac{\partial c}{\partial y} = \nu\left(\frac{\partial^2 c}{\partial x^2}+\frac{\partial^2 c}{\partial y^2}\right) + \frac{1}{2\Delta t}\left(\Delta x^2 a(1-a)\frac{\partial^2 c}{\partial x^2} - \Delta x\Delta y ab\frac{\partial^2 c}{\partial x\partial y} + \Delta y^2 b(1-b)\frac{\partial^2 c}{\partial y^2}\right) + \mathrm{HOT}, \tag{9}$$

where HOT stands for higher-order terms, while the term

$$\frac{1}{2\Delta t}\left(\Delta x^2 a(1-a)\frac{\partial^2 c}{\partial x^2} - \Delta x\Delta y ab\frac{\partial^2 c}{\partial x\partial y} + \Delta y^2 b(1-b)\frac{\partial^2 c}{\partial y^2}\right) \tag{10}$$

represents the numerical diffusion introduced artificially by the use of upwind differences. The artificial diffusion term (10) is directionally dependent. Hence, in convection-dominated problems, not only will the artificial diffusion prevail over the physical one, but drastically different numerical predictions can be obtained due only to different spatial orientation of the computational grid.[7]

The numerical diffusion of an upwind method can be reduced or eliminated when a higher-order differencing formula is used (see e.g. References 8–10). Leith's method,[9] for example, introduces no artificial diffusion but may develop large oscillations near sharp fronts;[11] attenuated spurious oscillations can also be developed by the method proposed by Leonard.[10] Moreover, in convection-dominated problems a severe limitation on the time step is imposed by the restrictions on the Courant numbers, which are always required not to exceed unity.

EULERIAN–LAGRANGIAN METHODS

In order to improve the accuracy and stability of the finite difference method (2), let us consider again equation (1) in the Lagrangian form

$$\frac{\mathrm{d}c}{\mathrm{d}t} = \nu\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right), \tag{11}$$

where the substantial derivative $\mathrm{d}/\mathrm{d}t$ indicates that the time rate of change is calculated along the streamline defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = v. \tag{12}$$

A natural explicit discretization of equation (11) is simply given by

$$\frac{c_{i,j}^{k+1} - c_{i-a,j-b}^{k}}{\Delta t} = \nu\left(\frac{c_{i-a+1,j-b}^{k} - 2c_{i-a,j-b}^{k} + c_{i-a-1,j-b}^{k}}{\Delta x^2} + \frac{c_{i-a,j-b+1}^{k} - 2c_{i-a,j-b}^{k} + c_{i-a,j-b-1}^{k}}{\Delta y^2}\right). \tag{13}$$

It is interesting to observe the physical significance of (13). The new c-value at time t_{k+1} in (i,j) is related to the value of c at time t_k in $(i-a, j-b)$ which diffuses in a lapsed time Δt.[11-13] Thus (13) is not only a simple algorithm but also accounts correctly for both convection and diffusion. In general, however, a and b are not integers (see Figure 1); therefore $(i-a, j-b)$ is not a grid point and an interpolation formula must be used to define $c_{i-a,j-b}^{k}$ and its neighbours in equation (13). The accuracy, stability, numerical diffusion and spurious oscillations of (13) depend on the interpolation formula chosen.

The simplest interpolation that can be taken for calculating $c_{i-a,j-b}^{k}$ and its neighbours is bilinear interpolation over the four surrounding mesh points. Let $a = n + p$ and $b = m + q$, where n and m are the integer parts of a and b respectively and p and q are their decimal parts. Then $c_{i-a,j-b}^{k}$ is approximated by

$$c_{i-a,j-b}^{k} = (1-p)[(1-q)c_{i-n,j-m}^{k} + qc_{i-n,j-m-1}^{k}] + p[(1-q)c_{i-n-1,j-m}^{k} + qc_{i-n-1,j-m-1}^{k}]. \tag{14}$$

Hence the Eulerian–Lagrangian method (13) can be written in the form (2), where the operator F is now given by

$$Fc_{i,j}^{k} = c_{i-a,j-b}^{k} + \nu\Delta t\left(\frac{c_{i-a+1,j-b}^{k} - 2c_{i-a,j-b}^{k} + c_{i-a-1,j-b}^{k}}{\Delta x^2} + \frac{c_{i-a,j-b+1}^{k} - 2c_{i-a,j-b}^{k} + c_{i-a,j-b-1}^{k}}{\Delta y^2}\right). \tag{15}$$

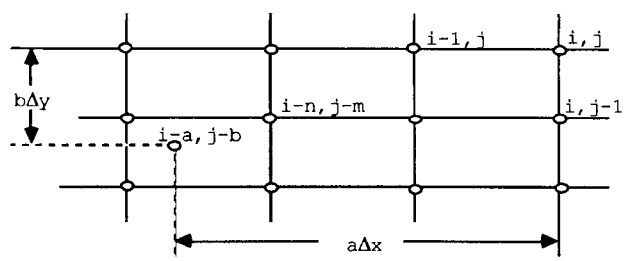

Figure 1

The amplification factor of F is given by

$$\begin{aligned} f = &[\cos(n\alpha) - \mathrm{I}\sin(n\alpha)][\cos(m\beta) - \mathrm{I}\sin(m\beta)] \\ &\times [1 - 2d_x - 2d_y + 2d_x\cos(\alpha) + 2d_y\cos(\beta)] \\ &\times [1 - p + p\cos(\alpha) - \mathrm{I}p\sin(\alpha)][1 - q + q\cos(\beta) - \mathrm{I}q\sin(\beta)]. \end{aligned} \tag{16}$$

From (16), after some simplifications, one gets

$$\begin{aligned} |f| \leqslant &|1 - 2d_x - 2d_y + 2d_x\cos(\alpha) + 2d_y\cos(\beta)| \\ &\times [|1 - p| + |p\cos(\alpha) - \mathrm{I}p\sin(\alpha)|] \\ &\times [|1 - q| + |q\cos(\beta) - \mathrm{I}q\sin(\beta)|]. \end{aligned} \tag{17}$$

Since p and q are positive numbers smaller than unity, the last two terms on the right-hand side of (17) have a modulus no greater than unity. Hence a necessary and sufficient condition to have $|f| \leqslant 1$ is given by

$$\Delta t \leqslant \left[2v\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right) \right]^{-1}. \tag{18}$$

Note that in convection-dominated problems inequality (18) is much less restrictive than either (5) or (8). This method, though explicit, becomes unconditionally stable in the particular case $v = 0$.

In order to examine the numerical diffusion introduced artificially by this scheme, each term in (13) is expanded in a Taylor series about $(i-a, j-b)$ to yield

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} = v\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} \right) + \frac{1}{2\Delta t}\left(\Delta x^2 p(1-p)\frac{\partial^2 c}{\partial x^2} + \Delta y^2 q(1-q)\frac{\partial^2 c}{\partial y^2} \right) + \text{HOT}. \tag{19}$$

Hence the term

$$\frac{1}{2\Delta t}\left(\Delta x^2 p(1-p)\frac{\partial^2 c}{\partial x^2} + \Delta y^2 q(1-q)\frac{\partial^2 c}{\partial y^2} \right) \tag{20}$$

represents the artificial diffusion, whose coefficients are now proportional to p and q rather than a and b. Consequently, since p and q are the decimal parts of a and b respectively, the artificial diffusion given by (20) is reduced with respect to the one given by (10). Further reduction can be obtained by increasing a and b; that is, either by increasing Δt or by reducing Δx and Δy.

Complete elimination of the numerical diffusion can be achieved by using a higher-order interpolation formula. For instance, if a biquadratic formula is used to interpolate $c^k_{i-a,\,j-b}$ and its neighbours, then by using n, m, p and q defined as above, one has

$$\begin{aligned} c^k_{i-a,j-b} = &\tfrac{1}{2}p(1+p)[\tfrac{1}{2}q(1+q)c^k_{i-n-1,j-m-1} + (1-q^2)c^k_{i-n-1,j-m} - \tfrac{1}{2}q(1-q)c^k_{i-n-1,j-m+1}] \\ &+ (1-p^2)[\tfrac{1}{2}q(1+q)c^k_{i-n,j-m-1} + (1-q^2)c^k_{i-n,j-m} - \tfrac{1}{2}q(1-q)c^k_{i-n,j-m+1}] \\ &- \tfrac{1}{2}p(1-p)[\tfrac{1}{2}q(1+q)c^k_{i-n+1,j-m-1} + (1-q^2)c^k_{i-n+1,j-m} \\ &- \tfrac{1}{2}q(1-q)c^k_{i-n+1,j-m+1}]. \end{aligned} \tag{21}$$

Thus the Eulerian–Lagrangian method (13) can be written in the form (2), where F is again defined by (15) but the interpolation formula is now (21) instead of (14). In this case the amplification factor of F is given by

$$\begin{aligned} f = &[\cos(n\alpha) - \mathrm{I}\sin(n\alpha)][\cos(m\beta) - \mathrm{I}\sin(m\beta)] \\ &\times [1 - 2d_x - 2d_y + 2d_x\cos(\alpha) + 2d_y\cos(\beta)] \\ &\times [1 - p^2 + p^2\cos(\alpha) - \mathrm{I}p\sin(\alpha)][1 - q^2 + q^2\cos(\beta) - \mathrm{I}q\sin(\beta)]. \end{aligned} \tag{22}$$

Hence

$$f^2 = [1-2d_x-2d_y+2d_x\cos(\alpha)+2d_y\cos(\beta)]^2 \times [1+p^2(p^2-1)(1-\cos(\alpha))^2][1+q^2(q^2-1)(1-\cos(\beta))^2]. \tag{23}$$

Thus, since p and q are smaller than unity, inequality (18) is the necessary and sufficient stability condition also when a biquadratic interpolation is used.

A Taylor series analysis of (21) shows that all second-order derivative terms cancel out; consequently, if a biquadratic interpolation is used, then the artificial diffusion of lowest order will not be introduced by (15). In this case, however, there may be some spurious oscillations which can be attenuated either by increasing Δt or by reducing Δx and Δy.

The Eulerian–Lagrangian methods described above also extend to the case when equation (1) is non-linear. In this case the determination of a and b requires that one solves equations (12) in which the right-hand sides are known only at time level t_k. Therefore it will be assumed that u and v do not vary over a time step and equations (12) will be integrated numerically backward from time level t_{k+1} to t_k by using, for instance, the Euler method. Specifically, the time step Δt is first divided into K equal parts of lengths $\tau = \Delta t/K$. Then, at each mesh point (i,j), equations (12) are discretized as follows:

$$\begin{aligned} x^{s+1} &= x^s - \tau u(x^s, y^s, t_{k+1} - s\tau, c(x^s, y^s, t_k)), & x^0 &= x_i, \\ y^{s+1} &= y^s - \tau v(x^s, y^s, t_{k+1} - s\tau, c(x^s, y^s, t_k)), & y^0 &= y_j, \end{aligned} \tag{24}$$

where $c(x^s, y^s, t_k)$ is defined by interpolation. The ending point (x^K, y^K) defines $(i-a, j-b)$. In so doing, the streamlines, which in general are not straight lines, are better approximated. This integration process is relatively fast, especially if performed on a vector machine. Indeed, the calculations in (24) are fully vectorizable, since for each s they are to be performed over the entire mesh configuration.[14] The present algorithm, however, becomes time-consuming when the diffusion dominates the convection terms, in which case there is no need to use the Eulerian–Lagrangian approach.

APPLICATION TO THE NAVIER–STOKES EQUATIONS

Consider the following two-dimensional Navier–Stokes equations in their primitive variables:

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \end{aligned} \tag{25}$$

where $u(x, y, t)$ and $v(x, y, t)$ are the velocity components in the x- and y-direction respectively, $p(x, y, t)$ is the pressure and ν is the kinematic viscosity coefficient, assumed to be constant and non-negative.

In order to solve equations (24) numerically, we introduce a spatial mesh which consists of rectangullar cells of width Δx and height Δy. The discrete horizontal velocity u is defined at the centre of each vertical side of a cell; the vertical velocity v is defined at the centre of each horizontal side; and the pressure p is defined at each cell centre (see Figure 2).

A general semi-implicit finite difference method for equations (24) is formulated as follows:[4,15]

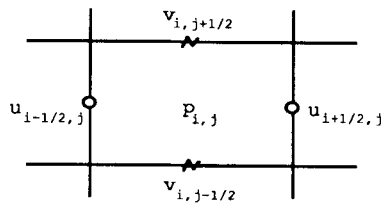

Figure 2

$$u_{i+1/2,j}^{k+1} = Fu_{i+1/2,j}^{k} - \frac{\Delta t}{\Delta x}\left(p_{i+1,j}^{k+1} - p_{i,j}^{k+1}\right), \tag{26}$$

$$v_{i,j+1/2}^{k+1} = Fv_{i,j+1/2}^{k} - \frac{\Delta t}{\Delta y}\left(p_{i,j+1}^{k+1} - p_{i,j}^{k+1}\right), \tag{27}$$

$$\frac{u_{i+1/2,j}^{k+1} - u_{i-1/2,j}^{k+1}}{\Delta x} + \frac{v_{i,j+1/2}^{k+1} - v_{i,j-1/2}^{k+1}}{\Delta y} = 0, \tag{28}$$

where F is a non-linear difference operator corresponding to the spatial discretization of the convective and viscous terms. A particular form for F can be chosen in a variety of ways. Thus the finite difference equations (26)–(28) form the base for a class of methods (pressure methods) for the numerical solution of the Navier–Stokes equations. The specification of F determines a particular method.

For any structure given to F the finite difference equations (26)–(28) constitute a linear system of equations with unknowns $u_{i+1/2,j}^{k+1}$, $v_{i,j+1/2}^{k+1}$ and $p_{i,j}^{k+1}$ over the entire cell configuration. This system has to be solved at each time step to determine recursively values of the field variables from given initial data. From a computational point of view, the determination of $u_{i+1/2,j}^{k+1}$, $v_{i,j+1/2}^{k+1}$ and $p_{i,j}^{k+1}$ can be accomplished efficiently if the corresponding linear system of equations (26)–(28) is first reduced to a smaller one having only the pressure $p_{i,j}^{k+1}$ as unknowns. Specifically, substitution of (26) and (27) into (28) yields

$$\frac{p_{i+1,j}^{k+1} - 2p_{i,j}^{k+1} + p_{i-1,j}^{k+1}}{(\Delta x)^2} + \frac{p_{i,j+1}^{k+1} - 2p_{i,j}^{k+1} + p_{i,j-1}^{k+1}}{(\Delta y)^2} = \frac{Fu_{i+1/2,j}^{k} - Fu_{i-1/2,j}^{k}}{\Delta t \Delta x} + \frac{Fv_{i,j+1/2}^{k} - Fv_{i,j-1/2}^{k}}{\Delta t \Delta y}. \tag{29}$$

This latter system can be solved, for example, by successive over-relaxation (see e.g. Reference 4). Once a value for the pressure has been found for each cell, the corresponding velocity field is determined uniquely from (26), (27). One then proceeds to the next time step.

Of course, if the calculations are to be performed using an array computer, a classical red/black ordering (or a more general multicolour ordering[16]) of the pressure equations is suggested, since such an ordering allows an efficient vectorization of each iteration. In this way, since the most expensive steps of the method (i.e. the evaluation of Fu and Fv, and the pressure iterations) are based on vector computations, a relatively high performance of this method is expected.

STABILITY OF THE METHOD

Owing to the difficulty in obtaining stability limits for non-linear finite difference equations, our stability analysis for equations (26)–(28) will be derived under the assumption that F is a linear

difference operator. Let $f=f(\Delta t, \alpha, \beta)$ be the complex amplification factor of the linearized operator F. A necessary and sufficient stability condition for the finite difference equations (26)–(28) is given by the following theorem.[15]

Theorem

The pressure method (26) is stable in the von Neumann sense if, and only if, $|f(\Delta t, \alpha, \beta)| \leqslant 1$ uniformly for every α and β.

Proof: Upon substitution of the Fourier modes to each field variable into the finite difference equations (26)–(28), one has

$$U^{k+1} = f U^k - \frac{\Delta t}{\Delta x}(e^{I\alpha/2} - e^{-I\alpha/2}) P^{k+1}, \tag{30}$$

$$V^{k+1} = f V^k - \frac{\Delta t}{\Delta y}(e^{I\beta/2} - e^{-I\beta/2}) P^{k+1}, \tag{31}$$

$$\frac{U^{k+1}}{\Delta x}(e^{I\alpha/2} - e^{-I\alpha/2}) + \frac{V^{k+1}}{\Delta y}(e^{I\beta/2} - e^{-I\beta/2}) = 0, \tag{32}$$

where U^k, V^k and P^k denote the amplitude functions of u, v and p at time level t_k. Equations (30)–(32) can also be written as

$$U^{k+1} = f U^k - 2I \frac{\Delta t}{\Delta x} \sin\left(\frac{\alpha}{2}\right) P^{k+1}, \tag{33}$$

$$V^{k+1} = f V^k - 2I \frac{\Delta t}{\Delta y} \sin\left(\frac{\beta}{2}\right) P^{k+1}, \tag{34}$$

$$\frac{U^{k+1}}{\Delta x} \sin\left(\frac{\alpha}{2}\right) + \frac{V^{k+1}}{\Delta y} \sin\left(\frac{\beta}{2}\right) = 0. \tag{35}$$

Substitution of (33) and (34) into (35) yields

$$f\left[\frac{U^k}{\Delta x} \sin\left(\frac{\alpha}{2}\right) + \frac{V^k}{\Delta y} \sin\left(\frac{\beta}{2}\right)\right] = 2I H \Delta t P^{k+1}, \tag{36}$$

where H is given by

$$H = \frac{\sin^2(\alpha/2)}{\Delta x^2} + \frac{\sin^2(\beta/2)}{\Delta y^2}. \tag{37}$$

By using (36), equations (33) and (34) become

$$\begin{aligned} U^{k+1} &= f U^k - f \frac{\sin(\alpha/2)}{H \Delta x}\left[\frac{U^k}{\Delta x} \sin\left(\frac{\alpha}{2}\right) + \frac{V^k}{\Delta y} \sin\left(\frac{\beta}{2}\right)\right], \\ V^{k+1} &= f V^k - f \frac{\sin(\beta/2)}{H \Delta y}\left[\frac{U^k}{\Delta x} \sin\left(\frac{\alpha}{2}\right) + \frac{V^k}{\Delta y} \sin\left(\frac{\beta}{2}\right)\right], \end{aligned} \tag{38}$$

or, in matrix notation,

$$\begin{bmatrix} U^{k+1} \\ V^{k+1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} U^k \\ V^k \end{bmatrix}, \tag{39}$$

where the amplification matrix **A** is given by

$$\mathbf{A} = \begin{bmatrix} f - f\dfrac{\sin^2(\alpha/2)}{H\Delta x^2} & -f\dfrac{\sin(\alpha/2)\sin(\beta/2)}{H\Delta x\Delta y} \\ -f\dfrac{\sin(\alpha/2)\sin(\beta/2)}{H\Delta x\Delta y} & f - f\dfrac{\sin^2(\beta/2)}{H\Delta y^2} \end{bmatrix}. \tag{40}$$

The eigenvalues of **A** are $\lambda_1 = 0$ and $\lambda_2 = f$ respectively. Hence, since **A** is a normal matrix, $|f| \leqslant 1$ is a necessary and sufficient stability condition for (26)–(28).

The above theorem implies that the pressure gradient and the velocity divergence discretization do not affect the stability of (26)–(28). The stability of this method is determined only by the linear difference operator F, that is, by the particular discretization chosen for the convective and viscous terms: any stable difference operator F for a convection–diffusion equation (1) also applies to (26)–(28) to define a stable method for the Navier–Stokes equations. Specifically, if F is given by (3), thg resulting algorithm is second-order accurate in space and will be stable when inequality (5) is satisfied; hence this method cannot apply for high-Reynolds flow. If the first-order upwind formula (6) is chosen to define F, the corresponding stability limit on the time step is given by inequality (8), but a large amount of artificial viscosity will be introduced at a high Reynolds number.

A drastic improvement in the stability and accuracy of method (26)–(28) can be achieved by using for F the Eulerian–Lagrangian form (15). The stability condition is simply given by (18) in both cases when a bilinear or a biquadratic interpolation is used. Inequality (18) becomes less restrictive as the Reynolds number increases; consequently, a larger time step or smaller space increment will be used to reduce the artificial viscosity in the first case, and the spurious oscillations in the second case.

Although inequality (18) on Δt is sufficient to assure the stability of the method, for accuracy we will impose a limitation on τ. Specifically, at each time step t_k the time subdivision τ will be taken small enough so that the corresponding Courant numbers will not exceed unity. That is,

$$\tau \leqslant \min\left[\frac{\Delta x}{\max\limits_{i,j} |u^k_{i+1/2,j}|}, \frac{\Delta y}{\max\limits_{i,j} |v^k_{i,j+1/2}|}\right]. \tag{41}$$

Note that when the bilinear interpolation is used, inequality (41) is also sufficient to assure that the streamlines approximated by (24) will not cross the solid boundaries. Assume, in fact, that the line $x = 0$ is a solid boundary and that (x^s, y^s), $x^s > 0$, is a point that lies inside a cell (i, j) of the computational domain. If $x^s \geqslant \Delta x$, then, by (24), inequality (41) implies $|x^{s+1} - x^s| \leqslant \Delta x$ and hence $x^{s+1} \geqslant 0$. If $x^s < \Delta x$, then cell (i, j) has the left vertical side on the boundary and hence $u^k_{i-1/2,j} = 0$ for all j. In this case from the first equation (24) one obtains

$$\begin{aligned} x^{s+1} &= x^s - \tau u^k(x^s, y^s) = x^s - \frac{\tau}{\Delta x}[(1 - x^s)u^k(0, y^s) + x^s u^k(\Delta x, y^s)] \\ &= x^s\left(1 - \frac{\tau}{\Delta x}u^k(\Delta x, y^s)\right) \geqslant 0. \end{aligned} \tag{42}$$

Hence in no case will the streamline approximated by (x^s, y^s), $s = 0, 1, 2, \ldots, K$, cross the solid boundary $x = 0$.

CAVITY FLOW PROBLEM

To emphasize the more important aspects of the Eulerian–Lagrangian method, we consider here the familiar time-dependent cavity flow problem. A square cavity whose top wall moves with a uniform velocity $u_T = 1$ is filled with an incompressible viscous fluid whose kinematic viscosity coefficient is $\nu = 0{\cdot}001$. At the initial time $t_0 = 0$ the fluid is at rest, that is $u(x, y, 0) = v(x, y, 0) = 0$.

The flow domain was divided into 50×50 finite difference cells of equal sides $\Delta x = \Delta y = 0{\cdot}02$ and the numerical solution of (24) has been generated at time $t_k = k\Delta t$. For such a small viscosity, centred differences (3) impose a limitation on Δt which cannot be larger than 0·002. On the other hand, the upwind differences (6) may allow the time step to be about $\Delta t = 0{\cdot}01$, but the artificial viscosity coefficients can be as large as 0·005, that is, five times higher than the physical viscosity. By using for F the Eulerian–Lagrangian form (15) with a time step $\Delta t = 0{\cdot}1$, a very accurate solution has been obtained in relatively short computer time. Figures 3 and 4 show the computed velocity direction at times $t = 10$ and $t = 100$ respectively (cf. Reference 1). At each mesh point the determination of a and b has been performed by using (24) with $K = 10$.

In a second example we consider the cavity flow problem described above but with the viscosity coefficient $\nu = 0{\cdot}0001$, which corresponds to the Reynolds number $Re = 10000$. In order to capture various secondary vortices, the square cavity was divided into 100×100 finite difference cells of sides $\Delta x = \Delta y = 0{\cdot}01$. Again, starting from rest and by using $\Delta t = 0{\cdot}1$ and $K = 10$, the

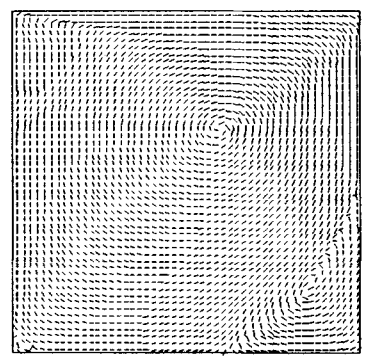

Figure 3. $Re = 1000$, $t = 10$

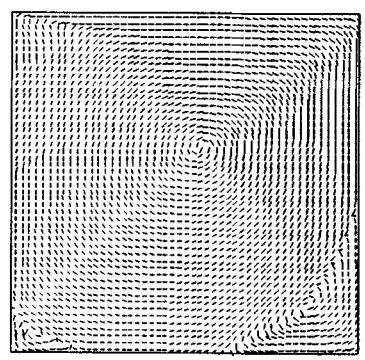

Figure 4. $Re = 1000$, $t = 100$

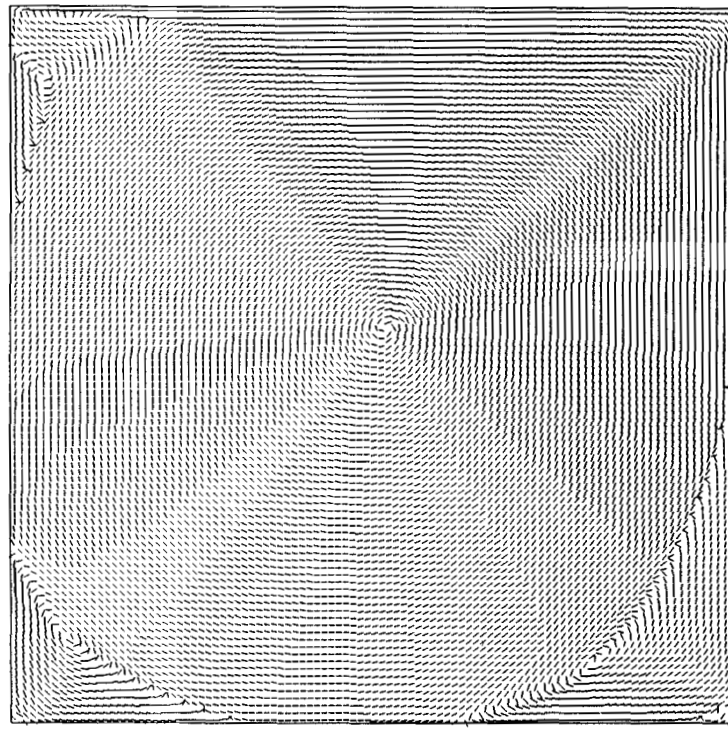

Figure 5. $Re = 10000$, $t = 100$

numerical solution at time $t = 100$ was obtained in only 3·5 min of CPU time on a CRAY XMP-48. Figure 5 shows the computed velocity directions at $t = 100$. These compare relatively well with the very accurate steady-state results reported in Reference 6. Here the linear system of pressure equations was solved at each time step using SOR with a red/black ordering. Although the specific FORTRAN program used to run the above example, for clarity, did not use diagonal storage of matrices, each internal loop was naturally vectorized by the Cray Fortran Translator, and high performance was achieved. For comparison, a run with the vectorization inhibited indicated that this algorithm becomes over four times slower, while the same code on a VAX 750 used 16 h and 18 min of CPU time to run the same problem.

CONCLUSIONS

A Eulerian–Lagrangian finite difference method for the Navier–Stokes equations has been analysed. At high Reynolds numbers the stability condition becomes less restrictive and the numerical viscosity, introduced artificially by this scheme, is demonstrated to be reduced at large time steps. Thus the method permits the use of larger time steps with corresponding improvements in both efficiency and accuracy. The numerical algorithm is so devised that it can be easily vectorized for efficient use on modern high-speed array computers. Further improvements in the accuracy of the present method can be achieved when it is used in combination with a multigrid technique.

REFERENCES

1. U. Ghia, K. N. Ghia and C. T. Shin, 'High-*Re* solutions for incompressible flow using the Navier–Stokes equations and a multigrid method', *J. Comput. Phys.*, **48,** (1982).
2. P. M. Gresho, S. T. Chan, R. L. Lee and C. D. Upson, 'A modified finite element method for solving the time-dependent, incompressible Navier–Stokes equations', *Int. j. numer. methods fluids*, **4,** (1984).
3. S. V. Patankar, *Numerical Heat Transfer and Fluid Flow*, Hemisphere, New York, 1980.
4. U. Bulgarelli, V. Casulli and D. Greenspan, *Pressure Methods for the Numerical Solution of Free Surface Fluid Flows*, Pineridge Press, Swansea, 1984.
5. F. H. Harlow and F. E. Welch, 'Numerical calculation of time-dependent viscous incompressible flow', *Phys. Fluids*, **8,** (1965).
6. A. C. Hindmarsh, P. M. Gresho and D. F. Griffiths, 'The stability of explicit Euler time-integration for certain finite difference approximations of the multidimensional advection–diffusion equation', *Int. j. numer. methods fluids*, **4,** (1984).
7. R. E. Ewing, *The Mathematics of Reservoir Simulation; SIAM Series Frontiers in Applied Mathematics, Vol. 1*, SIAM, Philadelphia, 1983.
8. JP. Huffenus and D. Khaletzky, 'The Lagrangian approach of advective term treatment and its application to the solution of Navier–Stokes equations', *Int. j. numer. methods fluids*, **1,** (1981).
9. C. E. Leith, 'Numerical simulation of the Earth's atmosphere', *Methods of Computational Physics, Vol. 4*, 1965.
10. B. P. Leonard, 'A stable and accurate convective modelling procedure based on quadratic upstream interpolation', *Comput. Methods Appl. Mech. Eng.*, **19,** (1979).
11. V. Casulli, 'Eulerian–Lagrangian methods for hyperbolic and convection dominated parabolic problems', in *Computational Methods for Nonlinear Problems*, Pineridge Press, Swansea, 1987.
12. R. W. MacCormack, 'An efficient explicit–implicit-characteristic method for solving the compressible Navier–Stokes equations', *SIAM–AMS Proc.*, **11,** (1978).
13. S. P. Neuman, 'A Eulerian–Lagrangian numerical scheme for the dispersion–convection equation using conjugate space–time grids', *J. Comput. Phys.*, **41,** (1981).
14. D. L. Book (ed.), *Finite-Difference Techniques for Vectorized Fluid Dynamic Calculations*, Springer-Verlag, New York, 1981.
15. V. Casulli, 'On the stability of the pressure method for the Navier–Stokes and Euler equations', *Proc. 3rd Int. Conf. on Boundary and Interior Layers*, Boole Press, Dublin, 1984.
16. E. L. Poole and J. M. Ortega, 'Multicolor ICCG methods for vector computers', *SIAM J. Numer. Anal.*, **24,** (1987).